Lecture Notes for Chapter 4

International Financial Markets and Institutions

Chapter 4

The forward market for foreign exchange

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Road Map

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4.8 Main Issues

- Implied risk-neutral probabilities
  - Binomial trees
  - Arrow-Debreu securities
  - Using Arrow-Debreu securities to price an option

4.9 Implied risk-neutral probabilities

We shall try and find out how to find the risk-neutral probability distribution of the spot rate at a given date.

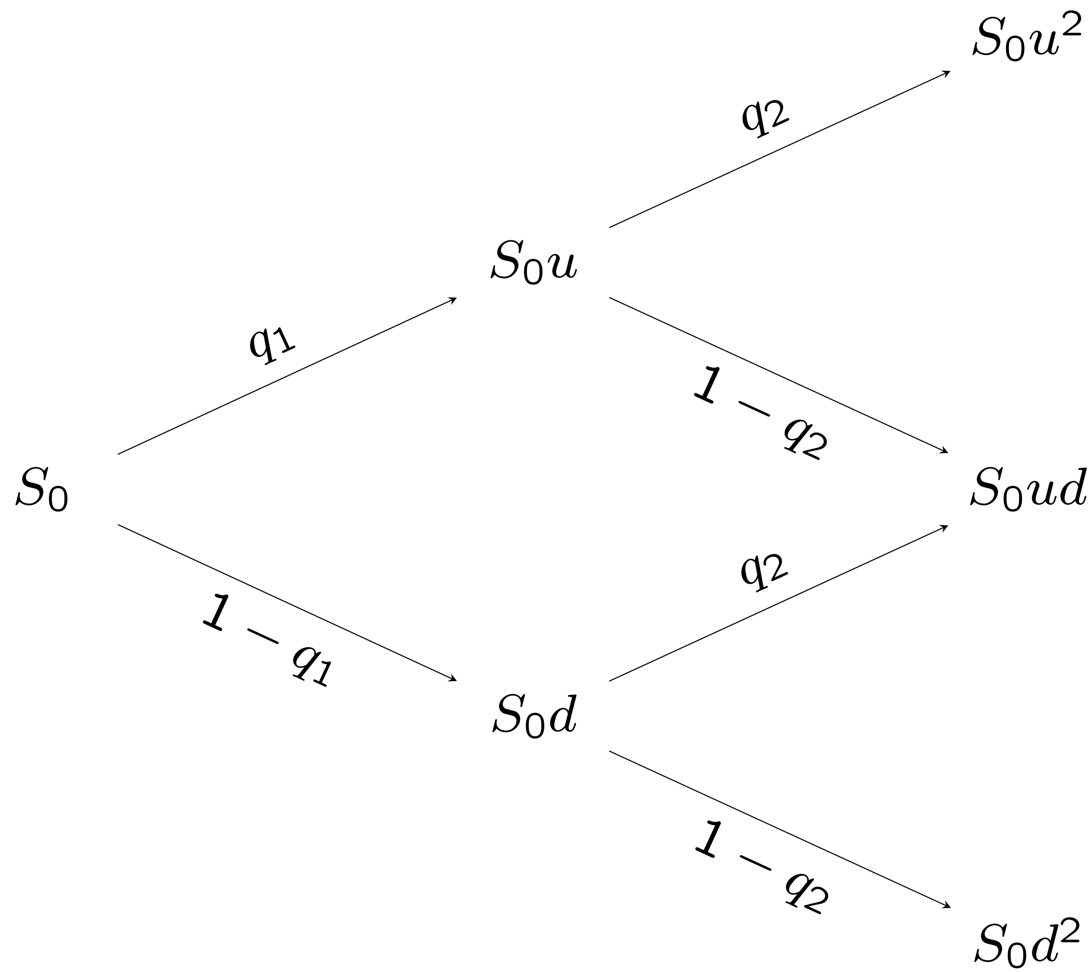
Why would we want to do this?

We will need to make some assumptions about how spot rates evolve over time.

4.9.1 Binomial tree

We shall assume spot rates can either move up or down at each date: **binomial tree**.

We shall assume the tree **recombines**.


S_0u^2
q_2
S_0u
q_1
$1 - q_2$
S_0
S_0ud
q_2
$1 - q_1$
S_0d
$1 - q_2$
S_0d^2

The **nodes** in the tree are often called **states**.

By assuming the spot rate 'lives' on a recombining binomial tree, we making very strong assumptions about how the spot prices evolves over time.

The tree structure also tells us how information about future spot rates changes over time.

- Date 0: we know that the spot rate at date 2 can take 3 possible values
- Date 1: we know the spot rate at date 2 can take 2 possible values

q_1 is the risk-neutral probability of the spot rate moving up at date 0

q_2 is the risk-neutral probability of the spot rate moving up at date 1

When the spot rate moves up it increases by a factor of $u > 1$. When it moves down the spot rate is multiplied by a factor of $d = \frac{1}{u} < 1$

We can see that

$$F_{0,1} = E_0^{\mathbb{Q}}[\tilde{S}_1] = q_1 S_0 u + (1 - q_1) S_0 d \tag{4.1}$$

$$\begin{aligned} F_{0,2} = E_0^{\mathbb{Q}}[\tilde{S}_2] &= q_1 q_2 S_0 u^2 + [q_1(1 - q_2) + (1 - q_1) q_2] S_0 ud \\ &+ (1 - q_1)(1 - q_2) S_0 d^2 \end{aligned} \tag{4.2}$$

Suppose we know that the current $\frac{CAD}{GBP}$ spot rate is 1.6, while the 1M and 2M forward rates are 1.601 and 1.603, respectively. Then, if we assume $u = 1.001$, we have

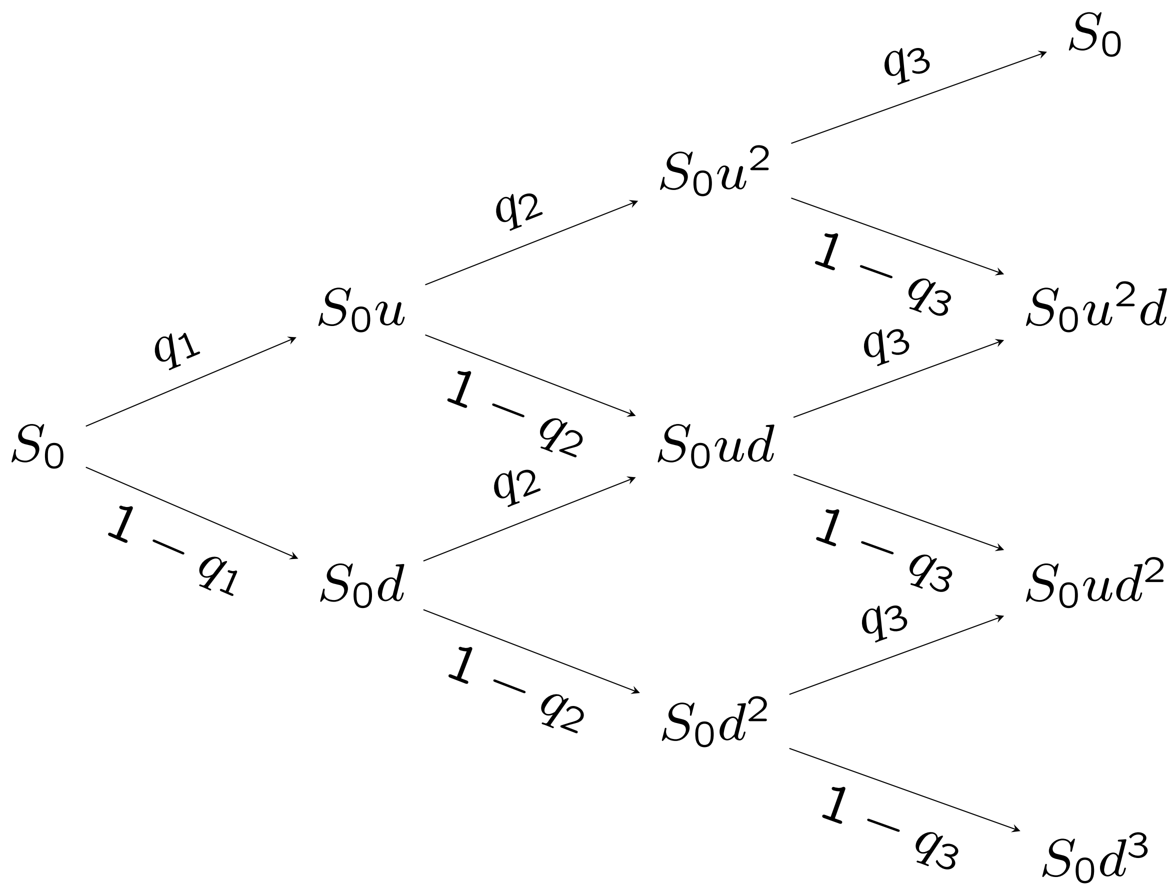
$$1.601 = 1.6[q_1 1.001 + (1 - q_1) 1.001^{-1}] \tag{4.3}$$

$$\begin{aligned} 1.603 = 1.6[&q_1 q_2 (1.001)^2 + [q_1(1 - q_2) + (1 - q_1) q_2] \\ &+ (1 - q_1)(1 - q_2)(1.001)^{-2}] \end{aligned} \tag{4.4}$$

Solve the first equation to find q_1

Solve the second equation to find q_2

Can go further than this? Yes!


S_0
q_1
$1-q_1$
S_0u
S_0d
q_2
$1-q_2$
q_2
$1-q_2$
S_0u^2
S_0ud
S_0d^2
q_3
$1-q_3$
q_3
$1-q_3$
q_3
$1-q_3$
S_0
S_0u^2d
S_0ud^2
S_0d^3

We can see that

$$F_{0,1} = E_0^{\mathbb{Q}}[\tilde{S}_1] = q_1 S_0 u + (1 - q_1) S_0 d \tag{4.5}$$

$$\begin{aligned} F_{0,2} &= E_0^{\mathbb{Q}}[\tilde{S}_2] = q_1 q_2 S_0 u^2 + [q_1(1 - q_2) + (1 - q_1) q_2] S_0 u d \\ &+ (1 - q_1)(1 - q_2) S_0 d^2 \end{aligned} \tag{4.6}$$

$$\begin{aligned} F_{0,3} &= E_0^{\mathbb{Q}}[\tilde{S}_2] = q_1 q_2 q_3 S_0 u^3 \\ &+ [q_1 q_2 (1 - q_3) + q_1 (1 - q_2) q_3 + (1 - q_1) q_2 q_3] \, S_0 u^2 d \\ &+ [q_1(1 - q_2)(1 - q_3) + (1 - q_1)(1 - q_2) q_3 + q_1 q_2 (1 - q_3)] \, S_0 u d^2 \\ &+ (1 - q_1)(1 - q_2)(1 - q_3) S_0 d^3 \end{aligned} \tag{4.7}$$

The only hard part is getting the coefficients correct.

One we have the risk-neutral probabilities on the tree for the spot rate, what can we do with them?

We can price claims which are perfectly correlated with spot rates.

4.9.2 Arrow-Debreu Claims on the Spot Rate

The first claims we shall look at only pay out 1 HC in a particular node. These are domestic **Arrow-Debreu** claims. We can also look at foreign **Arrow-Debreu** claims, which pay out 1 FC in a particular node.

Can think of Arrow-Debreu claims which are not contingent on the spot rate. The general definition of an Arrow-Debreu claim is

Definition 4.1

An **Arrow-Debreu** claim is a security which pays out one unit of some currency or good when a particular event occurs and zero otherwise.

Consider an event which is observed at date T but not before. Denote it via A_T.

The date-t HC price of the Arrow-Debreu claim which pays of 1 HC if event A_T occurs is

$$d_t^{HC}(A_T). \tag{4.8}$$

The date-t HC price of the Arrow-Debreu claim which pays 1 FC if event A_T occurs is

$$d_t^{*,HC}(A_T). \tag{4.9}$$

Arrow-Debreu claims are useful – once we have priced them, can price all claims (however complex) with payoffs that depend on events covered by the set of Arrow-Debreu claims.

- What is the date 0 price of the claim which pays 1 HC when the spot rate at date 1 equals $S_0 u$ and zero otherwise?

$$d_0^{HC}(\tilde{S}_1 = S_0 u) = HC \frac{q_1}{1 + r_{0,1}} \tag{4.10}$$

- To price the claim we use the risk-neutral probability q_1 and we need to know the effective risk-free rate between dates 0 and 1. We can obtain this risk-free rate from the date-0 price of a risk-free bond which pays 1 HC at date 1:

$$B_{0,1} = HC \frac{1}{1 + r_{0,1}} \tag{4.11}$$

Note that

$$d_0^{HC}(\tilde{S}_1 = S_0 u) = HCB_{0,1}q_1 = HC\frac{q_1}{1 + r_{0,1}} \tag{4.12}$$

- What is the date 0 price in units of HC of the claim which pays 1 FC when the spot rate at date 1 equals $S_0 u$ and zero otherwise?

$$d_0^{*,HC}(\tilde{S}_1 = S_0 u) = \frac{HC}{FC} S_0 \times 1FC \frac{q_1}{1 + r_{0,1}^*} = \frac{HC}{FC} F_{0,1} \times 1FC \frac{q_1}{1 + r_{0,1}} \tag{4.13}$$

- To price domestic (foreign) Arrow-Debreu claims contingent on the spot rate we need one of the following:
  - domestic and foreign risk-free bond prices for each date
  - domestic risk-free bond prices for each date and forward prices for each date
  - foreign risk-free bond prices for each date and forward prices for each date

- Can you see why the above is true?

4.9.3 Using Arrow-Debreu securities to price an option

Consider a European call option giving the right to buy 1 FC at date 1 for strike price of $K \frac{HC}{FC}$, where $S_0 d < K < S_0 u$.

- The price of this option is given by

$$HC \frac{q_1 (S_0 u - K)}{1 + r_{0,1}} = \underbrace{HC \frac{q_1}{1 + r_{0,1}}}_{\text{Arrow-Debreu price, } d_0^{HC}(\tilde{S}_1 = S_0 u)} \times (S_0 u - K) \quad (4.14)$$

Now consider a European call option giving the right to buy 1 FC at date 2 for strike price of $K \frac{HC}{FC}$, where $S_0 d < K < S_0 u$.